LIE ALGEBRAS OF DIFFERENTIAL OPERATORS AND PARTIAL INTEGRABILITY[§]

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Abstract

This paper surveys recent work on Lie algebras of differential operators and their application to the construction of quasi-exactly solvable Schrödinger operators.

1 Introduction

Lie-algebraic and Lie group theoretic methods have played a significant role in the development of quantum mechanics since its inception. In the classical applications, the Lie group appears as a symmetry group of the Hamiltonian operator, and the associated representation theory provides an algebraic means for computing the spectrum. Of particular importance are the exactly solvable problems, such as the harmonic oscillator or the hydrogen atom, whose point spectrum can be completely determined using purely algebraic methods. The fundamental concept of a "spectrum generating algebra" was introduced by Arima and Iachello, [6], [7], to study nuclear physics problems, and subsequently, by Iachello, Alhassid, Gürsey, Levine, Wu and their collaborators, was also successfully applied to molecular dynamics and spectroscopy, [23], [26], and scattering theory, [3], [4], [5]. The Schrödinger operators amenable to the algebraic approach assume a *Lie-algebraic form,* meaning that they belong to the universal enveloping algebra of the spectrum generating algebra. Lie-algebraic operators reappeared in the discovery of Turbiner, Shifman, Ushveridze, and their collaborators, [31], [33], [34], [40], of a new class of physically significant spectral problems, which they named *quasi-exactly solvable,*

having the property that a (finite) subset of the point spectrum can be determined using purely algebraic methods. This is an immediate consequence of the additional requirement that the hidden symmetry algebra preserve a finite-dimensional representation space consisting of smooth wave functions. In this case, the Hamiltonian restricts to a linear transformation on the representation space, and hence the associated eigenvalues can be computed by purely algebraic methods. Connections with conformal field theory, [21], [29], [32], [41], and the theory of orthogonal polynomials, [35], [36], [37], lend additional impetus for the study of such problems.

In higher dimensions, much less is known than in the one-dimensional case; in fact, only a few special examples of quasi-exactly solvable problems in two dimensions have appeared in the literature to date, [33], [18]. Complete lists of finite-dimensional Lie algebras of differential operators are known in two (complex) dimensions; there are essentially 24 different classes, some depending on parameters. The quasi-exactly solvable condition imposes a remarkable quantization constraint on the cohomology parameters classifying these Lie algebras. This phenomenon of the "quantization of cohomology" has recently been given an algebro-geometric interpretation, [12]. Any of the resulting quasi-exactly solvable Lie algebras of differential operators can be used to construct new examples of two-dimensional quasi-exactly solvable spectral problems. An additional complication is that, in higher dimensions, not every elliptic second-order differential operator is equivalent to a Schrödinger operator (i.e., minus Laplacian plus potential), so not every Lie-algebraic operator can be assigned an immediate physical meaning. The resulting "closure conditions" are quite complicated to solve, and so the problem of completely classifying quasi-exactly solvable Schrödinger operators in two dimensions appears to be too difficult to solve in full generality. A variety of interesting examples are given in [18], and we present a few particular cases of interest here.

The above ideas, originally introduced for scalar Hamiltonians describing spinless particles, can be generalized to include spin. The first step in this direction was taken by Shifman and Turbiner, [33], using the fact that a Hamiltonian for a spin 1/2 particle in d spatial dimensions can be constructed from a Lie superalgebra of first order differential operators in d ordinary (commuting) variables and one Grassmann (anticommuting) variable. Alternatively, [1], 2×2 matrices (or $N \times N$ matrices for particles of arbitrary spin, [2]) can be used to represent the Grassmann variable. In contrast with the scalar case, very few examples of matrix quasi-exactly solvable Schrödinger operators have been found thus far, [33]. There are important conceptual reasons for this fact. First of all, in the matrix case Lie superalgebras of matrix differential operators come naturally into play, while in the scalar case only Lie algebras need be considered. Secondly, as we shall see in Section 5, every scalar second order differential operator in one dimension can be transformed into a Schrödinger operator of the form $-\partial_x^2 + V(x)$ by a suitable change of the independent variable x and a local rescaling of the wave function. For matrix differential operators, on the other hand, the analogue of this result — $V(x)$ being now a Hermitian matrix of smooth functions — is no longer true unless the operator satisfies quite stringent conditions.

In the last section of this paper, we will discuss the characterization of the class of quasi-exactly solvable matrix differential operators preserving a finite-dimensional space

of wave functions with polynomial components (cf. [38], [1], [2]). For the important particular case of spin 1/2 particles, we will give necessary and sufficient conditions for a quasi-exactly solvable operator to be equivalent to a non-trivial Schrödinger operator. A suitable simplification of these conditions will then be used to construct new examples of multi-parameter quasi-exactly solvable spin 1/2 Hamiltonians in one dimension.

2 Quasi-Exactly Solvable Schrödinger Operators

Let M denote an open subset of Euclidean space $\mathbb{R}^d$ with coordinates $x = (x^1, \ldots, x^d)$. The time-independent Schrödinger equation for a differential operator $\mathcal{H}$ is the eigenvalue problem

$$\mathcal{H} \cdot \psi = \lambda \psi. \tag{1}$$

In the quantum mechanical interpretation, $\mathcal{H}$ is a (self-adjoint) second-order differential operator, which plays the role of the quantum "Hamiltonian" of the system. A nonzero wave function $\psi(x)$ is called em normalizable if it is square integrable, i.e., lies in the Hilbert space $\mathrm{L}^2(\mathbb{R}^d)$, and so represents a physical bound state of the quantum mechanical system, the corresponding eigenvalue determining the associated energy level. Complete explicit lists of eigenvalues and eigenfunctions are known for only a handful of classical "exactly solvable" operators, such as the harmonic oscillator. For the vast majority of quantum mechanical problems, the spectrum can, at best, only be approximated by numerical computation. The quasi-exactly solvable systems occupy an important intermediate station, in that a finite part of the spectrum can be computed by purely algebraic means.

To describe the general form of a quasi-exactly solvable operator, we begin with a finite-dimensional Lie algebra $\mathfrak{g}$ spanned by r linearly independent first-order differential operators

$$J^a = \sum_{i=1}^d \xi^{ai}(x) \, \frac{\partial}{\partial x^i} + \eta^a(x), \quad a = 1, \ldots, r, \tag{2}$$

whose coefficients ξ^{ai}, η^a are smooth functions of x. Note that each differential operator is a sum, $J^a = \mathbf{v}^a + \eta^a$, of a *vector field* $\mathbf{v}^a = \sum \xi^{ai} \partial / \partial x^i$ (which may be zero) and a *multiplication operator* η^a.

A differential operator is said to be *Lie-algebraic* if it lies in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$, meaning that it can be expressed as a polynomial in the operators J^a. In particular, a second-order differential operator T is Lie-algebraic if it can be written as a quadratic combination

$$-T = \sum_{a,b} c_{ab} J^a J^b + \sum_a c_a J^a + c_0, \tag{3}$$

for certain constants c_{ab}, c_a, c_0. (The minus sign in front of the Hamiltonian is taken for later convenience.) Note that the commutator $[J^c, T]$ of the Hamiltonian with any generator of $\mathfrak{g}$, while still of the same Lie-algebraic form, is not required to vanish. Therefore, the *hidden symmetry algebra* $\mathfrak{g}$ is not a symmetry algebra in the traditional sense.

Lie-algebraic operators appeared in the early work of Iachello, Levine, Alhassid, Gürsey and collaborators in the algebraic approach to scattering theory, [3], [4], [5], [26]. The condition of quasi-exact solvability imposes an additional constraint on the Lie algebra and hence on the type of operators which are allowed. A Lie algebra of first order differential operators $\mathfrak{g}$ will be called *quasi-exactly solvable* if it possesses a finite-dimensional representation space (or module) $\mathcal{N} \subset C^\infty$ consisting of smooth functions; this means that if $\psi \in \mathcal{N}$ and $J^a \in \mathfrak{g}$, then $J^a(\psi) \in \mathcal{N}$. A differential operator T is called *quasi-exactly solvable* if it lies in the universal enveloping algebra of a quasi-exactly solvable Lie algebra of differential operators. Clearly, the module $\mathcal{N}$ is an invariant space for T, i.e., $T(\mathcal{N}) \subset \mathcal{N}$, and hence T restricts to a linear matrix operator on $\mathcal{N}$. We will call the eigenvalues and corresponding eigenfunctions for the restriction $T|\mathcal{N}$ *algebraic,* since they can be computed by algebraic methods for matrix eigenvalue problems. (This does not mean that these functions are necessarily algebraic in the traditional pure mathematical sense.) Note that the number of such algebraic eigenvalues and eigenfunctions equals the dimension of $\mathcal{N}$. So far we have not imposed any normalizability conditions on the algebraic eigenfunctions, but, if they are normalizable, then the corresponding algebraic eigenvalues give part of the point spectrum of the differential operator.

It is of great interest to know when a given differential operator is in Lie algebraic or quasi-exactly solvable form. There is not, as far as we know, any direct test on the operator in question that will answer this in general. Consequently, the best approach to this problem is to effect a complete classification of such operators under an appropriate notion of equivalence. In order to classify Lie algebras of differential operators, and hence Lie algebraic and quasi-exactly solvable Schrödinger operators, we need to precisely specify the allowable equivalence transformations.

Definition 1 Two differential operators $T(x)$ and $\overline{T}(\bar{x})$ are *equivalent* if they can be mapped into each other by a combination of change of independent variable,

$$\bar{x} = \varphi(x), \tag{4}$$

and *gauge transformation*

$$\overline{T}(\bar{x}) = e^{\sigma(x)} \cdot T(x) \cdot e^{-\sigma(x)}. \tag{5}$$

The transformation (4)–(5) have two key properties. First, they respect the commutator between differential operators, and therefore preserve their Lie algebra structure. In particular, if $\mathfrak{g}$ is a finite-dimensional Lie algebra of first-order differential operators then the transformed algebra $\overline{\mathfrak{g}} = \{\overline{T} \mid T \in \mathfrak{g}\}$ is a Lie algebra isomorphic to $\mathfrak{g}$. Moreover, if $\mathcal{N}$ is a finite-dimensional $\mathfrak{g}$-module then $\overline{\mathcal{N}} = \{e^{\sigma(x)} f(x)|_{x = \varphi^{-1}(\bar{x})} \mid f \in \mathcal{N}\}$ is a finite-dimensional $\overline{\mathfrak{g}}$-module. In other words, if $\mathfrak{g}$ is quasi-exactly solvable so is $\overline{\mathfrak{g}}$. It immediately follows that the transformation (4)–(5) preserves the class of quasi-exactly solvable operators: in other words, if $T(x)$ is quasi-exactly solvable with respect to $\mathfrak{g}$ then the transformed operator $\overline{T}(\bar{x})$ will be quasi-exactly solvable with respect to the Lie algebra $\overline{\mathfrak{g}} \simeq \mathfrak{g}$.

Second, (4)–(5) preserve the spectral problem (1) associated to the differential operator T, so that if $\psi(x)$ is an eigenfunction of T with eigenvalue λ, then the transformed (or "gauged") function

$$\overline{\psi}(\bar{x}) = e^{\sigma(x)} \psi(x), \qquad \text{where} \qquad \bar{x} = \varphi(x), \tag{6}$$

is the corresponding eigenfunction of $\overline{T}$ having the same eigenvalue. Therefore, this notion of equivalence is completely adapted to the problem of classifying quasi-exactly solvable Schrödinger operators. The gauge factor $\mu(x) = e^{\sigma(x)}$ in (5) is *not* necessarily unimodular, i.e., $\sigma(x)$ is not restricted to be purely imaginary, and hence does not necessarily preserve the normalizability properties of the associated eigenfunctions. Therefore, the problem of normalizability of the resulting algebraic wave functions must be addressed.

Let us summarize the basic steps that are required in order to obtain a complete classification of quasi-exactly solvable operators and their algebraic eigenfunctions:

1. Classify finite-dimensional Lie algebras of differential operators.

2. Determine which Lie algebras are quasi-exactly solvable.

3. Solve the equivalence problem for differential operators.

4. Determine normalizability conditions.

5. Solve the associated matrix eigenvalue problem.

In this survey we will concentrate on the solution of the first three problems, referring the reader to [17] for the solution of the normalizability problem in the one-dimensional case.

3   Equivalence of Differential Operators

Consider a second-order linear differential operator

$$- T = \sum_{i,j=1}^{d} g^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} h^i(x) \frac{\partial}{\partial x^i} + k(x), \tag{7}$$

defined on an open subset $M \subset \mathbb{R}^d$. We are interested in studying the problem of when two such operators are equivalent under the combination of change of variables and gauge transformation (4), (5). Of particular importance is the question of when T is equivalent to a *Schrödinger operator,* which we take to mean an operator $\mathcal{H} = -\Delta + V(x)$, where Δ denotes either the flat space Laplacian or, more generally, the Laplace-Beltrami operator over a curved manifold. This definition of Schrödinger operator excludes the introduction of a magnetic field, which, however, can also be handled by these methods. As we shall see, there is an essential difference between one-dimensional and higher dimensional spaces in the solution to the equivalence problem for second-order differential operators, because in higher space dimensions not every second-order differential operator is locally equivalent to a Schrödinger operator $-\Delta + V(x)$, where Δ is the flat space Laplacian.

Explicit equivalence conditions were first found by É. Cotton, [8], in 1900. Since the symbol of a linear differential operator is invariant under coordinate transformations, we begin by assuming that the operator is elliptic, meaning that the symmetric matrix $\widehat{G}(x) = (g^{ij}(x))$ determined by the leading coefficients of $-T$ is positive-definite. Owing

to the induced transformation rules under the change of variables (7), we can interpret the inverse matrix $G(x) = \widehat{G}(x)^{-1} = (g_{ij}(x))$ as defining a Riemannian metric

$$ds^2 = \sum_{i,j=1}^{d} g_{ij}(x) dx^i dx^j, \tag{8}$$

on the subset $M \subset \mathbb{R}^d$. We will follow the usual tensor convention of raising and lowering indices with respect to the Riemannian metric (8). We rewrite the differential operator (7) in a more natural coordinate-independent form as

$$T = -\sum_{i,j=1}^{n} g^{ij}(\nabla_i - A_i)(\nabla_j - A_j) + V, \tag{9}$$

where ∇_i denotes covariant differentiation using the Levi-Civita connection associated to the metric ds^2. The vector $\mathbf{A}(x) = (A^1(x), \dots, A^d(x))$ can be thought of as a (generalized) magnetic vector potential; in view of its transformation properties, we define the associated *magnetic one-form*

$$A = \sum_{i=1}^{d} A_i(x)\, dx^i. \tag{10}$$

(Actually, to qualify as a physical vector potential, $\mathbf{A}$ must be purely imaginary and satisfy the stationary Maxwell equations, but we need not impose this additional physical constraint in our definition of the mathematical magnetic one-form (10).) The explicit formulas relating the covariant form (9) to the standard form (7) of the differential operator can be found in [18]. Each second-order elliptic operator then is uniquely specified by a metric, a magnetic one-form, and a potential function $V(x)$. In particular, if the magnetic form vanishes, so $A = 0$, then T has the form of a Schrödinger operator $T = -\Delta + V$, where Δ is the Laplace-Beltrami operator associated with the metric (8).

The application of a gauge transformation (5) does not affect the metric or the potential; however, the magnetic one-form is modified by an exact one-form: $A \mapsto A + d\sigma$. Consequently, the *magnetic two-form* $\Omega = dA$, whose coefficients represent the associated magnetic field, *is* unaffected by gauge transformations.

Theorem 2 *Two elliptic second-order differential operators T and $\overline{T}$ are (locally) equivalent under a change of variables $\bar{x} = \varphi(x)$ and gauge transformation (5) if and only if their metrics, their magnetic two-forms, and their potentials are mapped to each other*

$$\varphi^*(d\bar{s}^2) = ds^2, \qquad \varphi^*(\overline{\Omega}) = \Omega, \qquad \varphi^*(\overline{V}) = V. \tag{11}$$

(Here φ^* denotes the standard pull-back action of φ on differential forms; in particular, $\varphi^*(\overline{V}) = \overline{V} \circ \varphi$.)

In particular, an elliptic second-order differential operator is equivalent to a Schrödinger operator $-\Delta + V$ if and only if its magnetic one-form is closed: $dA = \Omega = 0$. Moreover, since the curvature tensor associated with the metric is invariant, T will be equivalent to a "flat" Schrödinger operator if and only if the metric ds^2 is flat, i.e., has vanishing Riemannian curvature tensor, and the magnetic one-form is exact.

In the one-dimensional case, every metric is automatically flat and all 1-forms are exact. Hence the previous theorem implies that every elliptic second-order differential operator is equivalent to a flat Schrödinger operator. See [14] for explicit formulas for the change of variables, the gauge factor and the potential.

4   Lie Algebras of Differential Operators

In this section, we summarize what is known about the classification problem for Lie algebras of first order differential operators. Any finite-dimensional Lie algebra $\mathfrak{g}$ of first order differential operators in $\mathbb{K}^d$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) has a basis of the form

$$J^1 = \mathbf{v}^1 + \eta^1(x), \ldots, J^r = \mathbf{v}^r + \eta^r(x),$$
$$J^{r+1} = f^1(x), \ldots, J^{r+s} = f^s(x), \tag{12}$$

cf. (2). Here $\mathbf{v}^1, \ldots, \mathbf{v}^r$ are linearly independent vector fields spanning an r-dimensional Lie algebra $\mathfrak{h}$. The functions $f^1(x), \ldots, f^s(x)$ define multiplication operators, and span an abelian subalgebra $\mathcal{F}$ of the full Lie algebra $\mathfrak{g}$. Since the commutator $[\mathbf{v}^i, f^j] = \mathbf{v}^i(f^j)$ is a multiplication operator, which must belong to $\mathfrak{g}$, we conclude that $\mathfrak{h}$ acts on $\mathcal{F}$, which is a finite-dimensional $\mathfrak{h}$-module (representation space) of smooth functions. The functions $\eta^a(x)$ must satisfy additional constraints in order that the operators (12) span a Lie algebra; we find

$$[\mathbf{v}^i + \eta^i, \mathbf{v}^j + \eta^j] = [\mathbf{v}^i, \mathbf{v}^j] + \mathbf{v}^i(\eta^j) - \mathbf{v}^j(\eta^i). \tag{13}$$

Now, since $\mathfrak{h}$ is a Lie algebra, $[\mathbf{v}^i, \mathbf{v}^j] = \sum_k c_k^{ij} \mathbf{v}^k$, where c_k^{ij} are the structure constants of $\mathfrak{h}$. Thus the above commutator will belong to $\mathfrak{g}$ if and only if

$$\mathbf{v}^i(\eta^j) - \mathbf{v}^j(\eta^i) - \sum_k c_k^{ij} \eta^k \in \mathcal{F}. \tag{14}$$

These conditions can be conveniently re-expressed using the basic theory of Lie algebra cohomology, [24]. Define the *one-cochain* F on the Lie algebra of vector fields $\mathfrak{h}$ as the linear map $F: \mathfrak{h} \to C^\infty \equiv C^\infty(\mathbb{K}^d)$ which satisfies $\langle F; \mathbf{v}^a \rangle = \eta^a$. Since we can add in any constant coefficient linear combination of the f^b's to the η^a's without changing the Lie algebra $\mathfrak{g}$, we should interpret the η^a's as lying in the quotient space $C^\infty/\mathcal{F}$, and hence regard F as a $C^\infty/\mathcal{F}$–valued cochain. In view of (13), the collection of differential operators (12) spans a Lie algebra if and only if the cochain F satisfies

$$\mathbf{v}\langle F; \mathbf{w} \rangle - \mathbf{w}\langle F; \mathbf{v} \rangle - \langle F; [\mathbf{v}, \mathbf{w}] \rangle \in \mathcal{F}, \qquad \forall \mathbf{v}, \mathbf{w} \in \mathfrak{h}. \tag{15}$$

The left hand side of (15) is just the evaluation $\langle \delta_1 F; \mathbf{v}, \mathbf{w} \rangle$ of the *coboundary* of the 1-cochain F, hence (15) expresses the fact that the cochain F must be a $C^\infty/\mathcal{F}$–valued *cocycle*, i.e., $\delta_1 F = 0$. A 1-cocycle is itself a coboundary, written $F = \delta_0 \sigma$ for some $\sigma(x) \in C^\infty$, if and only if $\langle F; \mathbf{v} \rangle = \mathbf{v}(\sigma)$ for all $\mathbf{v} \in \mathfrak{h}$, where $\mathbf{v}(\sigma)$ is considered as an element of $C^\infty/\mathcal{F}$. It can be shown that two cocycles will differ by a coboundary $\delta_0 \sigma$ if and only if the

corresponding Lie algebras are equivalent under the gauge transformation (5). Therefore two cocycles lying in the same *cohomology class* in the cohomology space $H^1(\mathfrak{h}, C^\infty/\mathcal{F}) = \text{Ker} \, \delta_1/\text{Im} \, \delta_0$ will give rise to equivalent Lie algebras of differential operators. In summary, then, we have the following fundamental characterization of Lie algebras of first order differential operators:

Theorem 3 *There is a one-to-one correspondence between equivalence classes of finite dimensional Lie algebras $\mathfrak{g}$ of first order differential operators on M and equivalence classes of triples $(\mathfrak{h}, \mathcal{F}, [F])$, where*

1. *$\mathfrak{h}$ is a finite-dimensional Lie algebra of vector fields,*

2. *$\mathcal{F} \subset C^\infty$ is a finite-dimensional $\mathfrak{h}$-module of functions,*

3. *$[F]$ is a cohomology class in $H^1(\mathfrak{h}, C^\infty/\mathcal{F})$.*

Based on Theorem 3, there are three basic steps required to classify finite dimensional Lie algebras of first order differential operators. First, one needs to classify the finite dimensional Lie algebras of vector fields $\mathfrak{h}$ up to changes of variables; this was done by Lie in one and two complex dimensions, [27], [28], and by the authors in two real dimensions, [16], under the regularity assumption that the Lie algebra has no singularities — not all vector fields in the Lie algebra vanish at a common point. Secondly, for each of these Lie algebras, one needs to classify all possible finite-dimensional $\mathfrak{h}$-modules $\mathcal{F}$ of C^∞ functions. Finally, for each of the modules $\mathcal{F}$, one needs to determine the first cohomology space $H^1(\mathfrak{h}, C^\infty/\mathcal{F})$. As the tables at the end of [15] indicate, the cohomology classes are parametrized by one or more continuous parameters or, in a few cases, smooth functions.

It is then a fairly straightforward matter to determine when a given Lie algebra satisfies the quasi-exact solvability condition that it admit a non-zero finite-dimensional module $\mathcal{N} \subset C^\infty$. Remarkably, in all known cases, the cohomology parameters are "quantized", the quasi-exact solvability requirement forcing them to assume at most a discrete set of distinct values. This intriguing phenomenon of "quantization of cohomology" has been geometrically explained in the maximal cases in terms of line bundles on complex surfaces in [12].

In one dimension, there is essentially only one quasi-exactly solvable Lie algebra of first order differential operators:

Theorem 4 *Every (non-singular) finite-dimensional quasi-exactly solvable Lie algebra of first order differential operators in one (real or complex) variable is locally equivalent to a subalgebra of one of the Lie algebras*

$$\widehat{\mathfrak{g}}_n = \text{Span}\{ \, \partial_x, \quad x\partial_x, \quad x^2\partial_x - n\,x, \quad 1 \, \} \simeq \mathfrak{gl}(2), \qquad (16)$$

where $n \in \mathbb{N}$. For $\widehat{\mathfrak{g}}_n$, the associated module $\mathcal{N} = \mathcal{P}_n$ consists of the polynomials of degree at most n.

Turning to the two-dimensional classification, a number of additional complications present themselves. First, as originally shown by Lie, there are many more equivalence classes of

finite-dimensional Lie algebras of vector fields, of arbitrarily large dimension and some-times depending on arbitrary functions as well. Moreover, the classification results in $\mathbb{R}^2$ and $\mathbb{C}^2$ are no longer the same; in fact, the real classification has been completed only very recently, [20]. Another complication is that the modules $\mathcal{F}$ for the vector field Lie algebras are no longer necessarily spanned by monomials, a fact that makes the de-termination of the cohomology considerably more difficult. Our classification results for finite-dimensional Lie algebras of differential operators in two real or complex variables, [13], [15], [20], are summarized in the Tables appearing, respectively, in [20] and [14]. The classification naturally distinguishes between the *imprimitive* Lie algebras, for which there exists an invariant foliation of the manifold, and the *primitive* algebras, having no such foliation. Each of the Lie algebras appearing in the complex classification has an obvious real counterpart, obtained by restricting the coordinates to be real. Moreover, as explained in [20], in such cases the associated real Lie algebras of differential opera-tors and finite-dimensional modules are readily obtained by restriction. As a matter of fact, every imprimitive real Lie algebra of vector fields in $\mathbb{R}^2$ is obtained by this simple procedure. In addition, there are precisely five primitive real Lie algebras of vector fields in two dimensions that are are not equivalent under a *real* change of coordinates to any of the Lie algebras obtained by straightforward restriction of the complex normal forms. The complete list of these additional real forms along with their canonical complexifica-tions appear in Table 4 of [20]. The maximal (complex) algebras, namely Case 1.4 of [20] ($\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$), Case 2.3 ($\mathfrak{sl}(3)$), and Case 1.11 ($\mathfrak{gl}(2) \ltimes \mathbb{R}^r$) play an important role in Turbiner's theory of differential equations in two dimensions with orthogonal polynomial solutions, [39]. Of the additional real normal forms, Case 4.2 ($\mathfrak{so}(3)$) was extensively used by Shifman and Turbiner, [33], while Case 4.5 ($\mathfrak{so}(3,1)$) is related to recent examples of quasi-exactly solvable Hamiltonians in two dimensions constructed by Zaslavskii, [42], [43].

5 Quasi-Exactly Solvable Operators on the Line

Let us now specialize to problems in one dimension. In view of Theorem 4, we let $n \in \mathbb{N}$ be a nonnegative integer, and consider the Lie algebra $\mathfrak{g}_n$ spanned by the differential operators

$$J^- = J_n^- = \frac{d}{dz}, \qquad J^0 = J_n^0 = z\frac{d}{dz} - \frac{n}{2}, \qquad J^+ = J_n^+ = z^2\frac{d}{dz} - nz, \qquad (17)$$

which satisfy the standard $\mathfrak{sl}(2)$ commutation relations. In this section, we shall use z instead of x for the "canonical coordinate", retaining x for the physical coordinate in which the operator takes Schrödinger form. Since $\mathfrak{g}_n$ differs from the Lie algebra $\widehat{\mathfrak{g}}_n$ in Theorem 4 only by the inclusion of constant functions, any Lie-algebraic operator (3) for the full algebra $\widehat{\mathfrak{g}}_n$ is automatically a Lie-algebraic operator for the subalgebra $\mathfrak{g}_n$. Therefore, in our analysis of Lie-algebraic differential operators we can, without loss of generality, concentrate on the Lie algebra $\mathfrak{g}_n$.

Using (17), it is readily seen that the most general second-order quasi-exactly solvable operator in one space dimension can be written in the canonical form

$$-T = P\frac{d^2}{dz^2} + \left\{Q - \frac{n-1}{2}P'\right\}\frac{d}{dz} + \left\{R - \frac{n}{2}Q' + \frac{n(n-1)}{12}P''\right\}, \tag{18}$$

where $P(z)$, $Q(z)$, and $R(z)$ are (general) polynomials of respective degrees $4, 2, 0$, whose explicit expression in terms of the constants c_{ab} and c_a can be found in [14]. Since the module $\mathcal{N}$ is the space $\mathcal{P}_n$ of polynomials of degree at most n, the algebraic eigenfunctions of (18) will, in the z-coordinates, just be polynomials $\chi_k(z) \in \mathcal{P}_n$. In terms of the standard basis $\nu_k(z) = z^k$, $k = 0, \ldots, n$, the restriction $T \mid \mathcal{P}_n$ takes the form of a pentadiagonal matrix. In summary, for a normalizable one-dimensional quasi-exactly solvable operator there are $n + 1$ algebraic eigenfunctions which, in the canonical z coordinates, are polynomials of degree at most n.

Specializing the solution to the equivalence problem given by Theorem 2 to the operator (18), we find (cf. [14]) that if $P(z) > 0$ the change of variables required to place the operator into physical (Schrödinger) form will, in general, be given by an elliptic integral

$$x = \varphi(z) = \int^z \frac{dy}{\sqrt{P(y)}}, \tag{19}$$

the corresponding gauge factor being

$$\mu(z) = P(z)^{-n/4} \exp\left\{\int^z \frac{Q(y)}{2P(y)}\,dy\right\}. \tag{20}$$

The potential is given by

$$V(x) = -\frac{n(n+2)\left(PP'' - \frac{3}{4}P'^2\right) + 3(n+1)(QP' - 2PQ') - 3Q^2}{12P} - R, \tag{21}$$

where the right hand side is evaluated at $z = \varphi^{-1}(x)$. In the physical coordinate, the associated algebraic wave functions will then take the form

$$\psi(x) = \mu\left(\varphi^{-1}(x)\right) \cdot \chi\left(\varphi^{-1}(x)\right), \tag{22}$$

where $\chi(z)$ is a polynomial of degree at most n.

The canonical form (18) of a quasi-exactly solvable differential operator is not unique, since there is a "residual" symmetry group which preserves the Lie algebra $\mathfrak{g}_n$. Not surprisingly, this group is $\mathrm{GL}(2, \mathbb{R})$, which acts on the (projective) line by linear fractional (Möbius) transformations

$$z \longmapsto w = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \det A = \Delta = \alpha\delta - \beta\gamma \neq 0. \tag{23}$$

To describe the induced action of the transformations (23) on the quasi-exactly solvable operators (18), we first recall the basic construction of the finite-dimensional irreducible rational representations of the general linear group $\mathrm{GL}(2, \mathbb{R})$.

Definition 5 Let $n \geq 0$, i be integers. The irreducible multiplier representation $\rho_{n,i}$ of $\mathrm{GL}(2,\mathbb{R})$ is defined on the space $\mathcal{P}_n$ of polynomials of degree at most n by the transformation rule

$$P(w) \mapsto \widehat{P}(z) = (\alpha\delta - \beta\gamma)^i \, (\gamma z + \delta)^n \, P\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right), \qquad P \in \mathcal{P}_n. \tag{24}$$

The multiplier representation $\rho_{n,i}$ has infinitesimal generators given by the differential operators (17) combined with the operator of multiplication by $n + 2i$ representing the diagonal subalgebra (center) of $\mathfrak{gl}(2,\mathbb{R})$. The action (23) induces an automorphism of the Lie algebra $\mathfrak{g}_n$, which is isomorphic to the representation $\rho_{2,-1}$, and, consequently, preserves the class of quasi-exactly solvable operators associated with the algebra $\mathfrak{g}_n$. Moreover, the corresponding gauge action

$$\widehat{T}(z) = (\gamma z + \delta)^n \cdot T(w) \cdot (\gamma z + \delta)^{-n}, \qquad w = \frac{\alpha z + \beta}{\gamma z + \delta} \tag{25}$$

will preserve the space of quasi-exactly solvable operators (18). Identifying the operator T with the corresponding quartic, quadratic and constant polynomials P, Q, R, we find that the action (25) of $\mathrm{GL}(2,\mathbb{R})$ on the space of quasi-exactly solvable second-order operators is isomorphic to the sum of three irreducible representations, $\rho_{4,-2} \oplus \rho_{2,-1} \oplus \rho_{0,0}$; the quartic $P(z)$ transforms according to $\rho_{4,-2}$, the quadratic $Q(z)$ according to $\rho_{2,-1}$, while R is constant. Finally, the associated module, which is just the space of polynomials $\mathcal{P}_n$, transforms according to the representation $\rho_{n,0}$.

Using the action of $\mathrm{GL}(2,\mathbb{R})$, we can place the gauged operator (18) into a simpler canonical form, based on the invariant-theoretic classification of canonical forms for real quartic polynomials, [22], [17].

Theorem 6 *Under the representation $\rho_{4,-2}$ of $\mathrm{GL}(2,\mathbb{R})$, every nonzero real quartic polynomial $P(z)$ is equivalent to one of the following canonical forms:*

$$\nu(1 - z^2)(1 - k^2 z^2), \qquad \nu(1 - z^2)(1 - k^2 + k^2 z^2), \qquad \nu(1 + z^2)(1 + k^2 z^2),$$
$$\nu(z^2 - 1), \quad \nu(z^2 + 1), \quad \nu z^2, \quad \nu(z^2 + 1)^2, \quad z, \quad 1, \tag{26}$$

where ν and $0 < k < 1$ are real constants.

The solution to the normalizability problem, which the interested reader can find in [17], begins with a detailed analysis of the elliptic integral (19). It is found, first of all, that the class of quasi-exactly solvable potentials naturally splits into two subclasses — the periodic potentials, which are never normalizable, and the non-periodic potentials, which are sometimes normalizable. Tedious but direct calculations based on (19), (20), (21), and (22), produce then the explicit change of variables, the potential, and the form of the algebraic eigenfunctions for the above normal forms in physical coordinates. Each of the classes of potentials is a linear combination of four elementary and/or elliptic functions, plus a constant which we absorb into the eigenvalue. The potentials naturally divide into two classes, which are listed in the following two Tables. In each case, the four coefficients

are not arbitrary, but satisfy a single complicated algebraic equation and one or more inequalities; see [17] for the details.

First, the periodic quasi-exactly solvable potentials correspond to the cases when the real roots of P (if any) are simple, of which there are five cases in (26). The explicit formulas for the corresponding potentials follow; they depend on two real parameters $\alpha > 0$ and $k \in (0, 1)$. In the first three cases, the corresponding potentials are written in terms of the standard Jacobi elliptic functions of modulus k, [9, vol. 2]. Also, as remarked above, the coefficients A, B, C, D are not arbitrary, although the explicit constraints are too complicated to write here.

Periodic Quasi-Exactly Solvable Potentials

1. $\operatorname{dn}^{-2}\alpha\,x(A\operatorname{sn}\alpha\,x + B) + \operatorname{cn}^{-2}\alpha\,x(C\operatorname{sn}\alpha\,x + D)$
2. $\operatorname{dn}^{-2}\alpha\,x(A\operatorname{cn}\alpha\,x + B) + \operatorname{sn}^{-2}\alpha\,x(C\operatorname{cn}\alpha\,x + D)$
3. $A\operatorname{cn}\alpha\,x\operatorname{sn}\alpha\,x + B\operatorname{cn}^2\alpha\,x + C\operatorname{dn}^{-2}\alpha\,x(\operatorname{cn}\alpha\,x\operatorname{sn}\alpha\,x + D\operatorname{cn}^2\alpha\,x)$
4. $A\sin^2\alpha\,x + B\sin\alpha\,x + C\tan\alpha\,x\sec\alpha\,x + D\sec^2\alpha\,x$
5. $A\cos 4\alpha\,x + B\cos 2\alpha\,x + C\sin 2\alpha\,x + D\sin 4\alpha\,x$

Note that the potentials in cases 1, 2 and 4 have singularities unless $C = D = 0$. In Cases 3 and 5, the potential has no singularities, reflecting the fact that in these cases $P(z)$ has no real roots. Case 3 includes the Lamé equation, [9, vol. 3], and Case 4 with $A = B = 0$ is the trigonometric Scarf potential.

The non-periodic potentials correspond to the cases with one or two multiple roots. The explicit formulas for the corresponding potentials follow (again, $\alpha > 0$ is a real constant).

Non-periodic Quasi-Exactly Solvable Potentials

1. $A\sinh^2\alpha\,x + B\sinh\alpha\,x + C\tanh\alpha\,x\operatorname{sech}\alpha\,x + D\operatorname{sech}^2\alpha\,x$
2. $A\cosh^2\alpha\,x + B\cosh\alpha\,x + C\coth\alpha\,x\operatorname{csch}\alpha\,x + D\operatorname{csch}^2\alpha\,x$
3. $Ae^{2\alpha\,x} + Be^{\alpha\,x} + Ce^{-\alpha\,x} + De^{-2\alpha\,x}$
4. $Ax^6 + Bx^4 + Cx^2 + \dfrac{D}{x^2}$
5. $Ax^4 + Bx^3 + Cx^2 + Dx$

In cases 2 and 4, the potential has a singularity at $x = 0$ unless $C + D = 0$ (Case 2) or $D = 0$ (Case 4). The nonsingular potentials in Case 4 are the anharmonic oscillator potentials discussed in detail in [34], [32]. The algebraic constraints satisfied by the coefficients are given in [17].

According to Turbiner, [34], a potential is *exactly solvable* if it does not explicitly depend on the discrete "spin" parameter n, since, in this case, one can find representation spaces of arbitrarily large dimension and thereby (if the algebraic eigenfunctions are

normalizable) produce infinitely many eigenvalues by algebraic methods. Note that since the gauge transformation (19), (20), can explicitly depend on n, exact solvability cannot be detected in the canonical coordinates, but depends on the final physical form of the operator. The exactly solvable nonperiodic potentials are characterized by the condition $A = B = 0$, and, in case 3, $C = D = 0$. In Case 2, there is an additional inequality, $|C| \leq D + \frac{1}{4}\alpha^2$, to be satisfied. The exactly solvable potentials include the (restricted) Pöschl–Teller and Scarf potentials (Case 1), the Rosen–Morse II potential (Case 2), the Morse potential (Case 3), the radial harmonic oscillator (provided $D = l(l+1)$, $l \in \mathbb{N}$) (Case 4), and the harmonic oscillator (Case 5).

Analysis of the explicit formulas for the eigenfunctions based on (22) yields a complete set of conditions for the normalizability of the non-periodic potentials, which can be written explicitly in terms of the coefficients of the quadratic polynomial $Q(z)$. It is then possible to deduce explicit, general normalizability conditions on the Lie-algebraic coefficients c_{ab} and c_a by using the fact that such conditions must be invariant under the action of the group $\mathrm{GL}(2,\mathbb{R})$. Therefore, normalizability conditions can be written in terms of the classical joint invariants and covariants of the pair of polynomials P, Q. See [17] for the details and a complete list of invariant normalizability conditions.

6 Two-Dimensional Problems

There are a number of additional difficulties in the two-dimensional problem which do not appear in the scalar case. First, there are several different classes of quasi-exactly solvable Lie algebras available. Even more important is the fact that, according to Theorem 2, there are nontrivial *closure conditions* which must be satisfied in order that the magnetic one-form associated with a given second-order differential operator be closed, and hence the operator be equivalent under a gauge transformation (5) to a Schrödinger operator. Unfortunately, in all but trivial cases, the closure conditions associated with a quasi-exactly solvable operator (3) corresponding to the generators of one of the quasi-exactly solvable Lie algebras on our list are *nonlinear algebraic equations* in the coefficients c_{ab}, c_a, c_0, and it appears to be impossible to determine their general solution. Nevertheless, there are useful simplifications of the general closure conditions which can be effectively used to generate large classes of planar quasi-exactly solvable and exactly solvable Schrödinger operators, both for flat space as well as curved metrics.

Suppose that the Lie algebra $\mathfrak{g}$ is spanned by linearly independent first-order differential operators as in (2). The *closure conditions* $dA = 0$ for a Lie-algebraic second-order differential operator (3) are then equivalent to the solvability of the system of partial differential equations

$$\sum_{a,b=1}^{r} c_{ab}\xi^{ai} \sum_{j=1}^{n} \left(\xi^{bj} \frac{\partial \tau}{\partial x^j} + \frac{\partial \xi^{bj}}{\partial x^j} \right) = \sum_{a=1}^{r} \xi^{ai} \left[2\sum_{b=1}^{r} c_{ab}\eta^b + c_a \right],$$
$$i = 1, \ldots, d, \qquad (27)$$

for a scalar function $\tau(x)$, given by $\tau = 2\sigma + \frac{1}{2}\log\det(g_{ij})$ in terms of the gauge factor e^σ required to place the operator in Schrödinger form. The closure conditions (27) are

extremely complicated to solve in full generality, but a useful subclass of solutions can be obtained from the *simplified closure conditions*

$$\sum_{i=1}^{n}\left(\xi^{ai}\frac{\partial\tau}{\partial x^i}+\frac{\partial\xi^{ai}}{\partial x^i}\right)-2\eta^a=k^a, \qquad a=1,\ldots,r. \tag{28}$$

where $k^1,\ldots,k^r$ are constants. Any solution $\tau(x)$ of equations (28) will generate an infinity of solutions to the full closure conditions (27), with c_{ab} arbitrary, and $c_a=\sum_b c_{ab}k^b$. The case $k^a=0$ and $\mathfrak{g}$ semi-simple was investigated in [29]. Although the simplified closure conditions can be explicitly solved for such Lie algebras, with the exception of $\mathfrak{so}(3)$, their solutions are found to generate quasi-exactly solvable Schrödinger operators that are *not* normalizable, and hence of limited use. Note that even when the simplified closure conditions do not have any acceptable solutions, the *full* closure conditions (27) may be compatible and may give rise to normalizable operators, as shown in [18] and [20]. Here we shall limit ourselves to a few examples.

Consider, in the first place, the Lie algebra $\mathfrak{g}\simeq\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$ spanned by the first-order differential operators

$$J^1=\partial_x, \quad J^2=\partial_y, \quad J^3=x\partial_x,$$
$$J^4=y\partial_y, \quad J^5=x^2\partial_x-nx, \quad J^6=y^2\partial_y-my, \tag{29}$$

where $n,m\in\mathbb{N}$. The particular choice

$$(c_{ab}) = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \tag{30}$$

$$(c_a) = \Big(0,0,-(1+4n),-(1+4m),0,0\Big), \qquad c_0=\tfrac{3}{4}+m^2+n^2, \tag{31}$$

of Lie-algebraic coefficients lead to a quasi-exactly solvable Hamiltonian with Riemannian metric

$$g^{11}=(1+x^2)(2+x^2), \quad g^{12}=(1+x^2)(1+y^2), \quad g^{22}=(1+y^2)(2+y^2), \tag{32}$$

which has complicated curvature, and potential

$$\begin{aligned} 4V &= -y^2-\frac{(1+2n)(3+2n)}{1+x^2}-\frac{(1+2m)(3+2m)}{1+y^2} \\ &\quad -\frac{17+12y^2-y^4+2xy(6+5y^2)}{3+x^2+y^2} \\ &\quad +\frac{5(3+2xy)(1+y^2)(2+y^2)}{(3+x^2+y^2)^2}. \end{aligned} \tag{33}$$

Consider next the $\mathfrak{so}(3,1)$ Lie algebra generated by the operators

$$J^1 = \partial_x, \quad J^2 = \partial_y, \quad J^3 = x\partial_x + y\partial_y, \quad J^4 = y\partial_x - x\partial_y,$$
$$J^5 = (x^2 - y^2)\partial_x + 2xy\partial_y - 2nx, \quad J^6 = 2xy\partial_x + (y^2 - x^2)\partial_y - 2ny, \qquad (34)$$

with $n \in \mathbb{N}$ a positive integer. Notice that this algebra, although of course inequivalent to any of the *real* normal forms in the classification of [20], is *complex*-equivalent to the $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ Lie algebra in the previous example under the complex change of variables $z = x + iy$, $w = x - iy$.

The second-order differential operator

$$-T = \alpha \left[(J^1)^2 + (J^2)^2 \right] + \beta (J^3)^2 + \gamma (J^4)^2$$
$$+ \lambda \left[(J^5)^2 + (J^6)^2 \right] + 4\beta(1 + 2n)^2 - 4\gamma \qquad (35)$$

satisfies the closure conditions, and is therefore equivalent to a Schrödinger operator on the manifold with contravariant metric tensor (g^{ij}) given by

$$
\begin{aligned}
g^{11} &= \alpha + \beta x^2 + \gamma y^2 + \lambda (x^2 + y^2)^2, \\
g^{12} &= (\beta - \gamma) x y, \\
g^{22} &= \alpha + \gamma x^2 + \beta y^2 + \lambda (x^2 + y^2)^2.
\end{aligned}
\qquad (36)
$$

The Gaussian curvature is

$$\kappa = \frac{(-\beta + 3\gamma)(\alpha^2 + \lambda^2 r^8) + 2(\beta\gamma + 4\alpha\lambda)r^2(\alpha + \lambda r^4) + 2\alpha\lambda(5\beta + \gamma)r^4}{(\alpha + \gamma r^2 + \lambda r^4)^2}, \qquad (37)$$

with $r^2 = x^2 + y^2$. If the parameters α, β, γ and λ are positive, then the metric is non-singular and positive definite for (x, y) ranging over all of $\mathbb{R}^2$. The fact that the closure conditions are satisfied guarantees the existence of a gauge factor μ such that $H = \mu T \mu^{-1}$ is a Schrödinger operator; see [20] for an explicit formula for the gauge factor, which is too complicated to be given here. The expression for the potential V is

$$
\begin{aligned}
4V \; = \; & \frac{16 \alpha \beta n (1 + n) + r^2 \left[\beta^2 (3 + 16 n + 16 n^2) - 4 \alpha \lambda (3 + 8 n + 4 n^2)\right]}{\alpha + \beta r^2 + \lambda r^4} \\
& + \frac{5 (\beta - \gamma) (4 \alpha \lambda - \gamma^2) + 3 \lambda (2 \beta \gamma - 3 \gamma^2 + 4 \alpha \lambda) r^2}{\lambda (\alpha + \gamma r^2 + \lambda r^4)} \\
& - \frac{5 (\beta - \gamma) (4 \alpha \lambda - \gamma^2) (\alpha + \gamma r^2)}{\lambda (\alpha + \gamma r^2 + \lambda r^4)^2}.
\end{aligned}
\qquad (38)
$$

Since the potential is a function of r only, it is natural to look for eigenfunctions of H which depend on r only. When this is done, it can be shown that one ends up with an effective Hamiltonian on the line which is an element of the enveloping algebra of the standard realization of $\mathfrak{sl}(2, \mathbb{R})$ in one dimension. Thus, no new quasi-exactly solvable one-dimensional potentials are obtained by reduction of the above quasi-exactly solvable $\mathfrak{so}(3,1)$ potential. This lends additional support to the observation that reduction of

two-dimensional quasi-exactly solvable Schrödinger operators does not lead to any new one-dimensional quasi-exactly solvable operators.

Non semi-simple Lie algebras can also yield interesting examples of quasi-exactly solvable potentials. Indeed, let $\mathfrak{g}$ be the Lie algebra spanned by the first-order differential operators

$$J^1 = \partial_x, \quad J^2 = \partial_y, \quad J^3 = x\partial_x, \quad J^4 = x\partial_y, \quad J^5 = y\partial_y, \tag{39}$$

and

$$J^6 = x^2\partial_x + (r-1)xy\partial_y - nx, \quad J^{6+i} = x^{i+1}\partial_y, \qquad i = 1, \ldots, r-2. \tag{40}$$

The module $\mathcal{N}$ is now spanned by the monomials $x^i y^j$ with $i + (r-1)j \leq n$. For $m \in \mathbb{N}$, $\alpha, \beta > 0$, the Schrödinger operator with metric

$$g^{11} = \alpha x^2 + \beta, \quad g^{12} = (1+m)\alpha xy, \quad g^{22} = (\alpha x^2 + \beta)^m + \alpha(1+m)^2 y^2, \tag{41}$$

and potential

$$V = -\frac{\lambda\alpha\beta(1+m)^2(\alpha x^2 + \beta)^m}{(\alpha x^2 + \beta)^{1+m} + \alpha\beta(1+m)^2 y^2}, \qquad m \leq r - 1 \neq 2(m+1), \tag{42}$$

is normalizable and quasi-exactly solvable with respect to $\mathfrak{g}$, provided that the parameter λ is large enough. The metric in this case has constant negative Gaussian curvature $\kappa = -\alpha$. Furthermore, since the potential V does not depend on the cohomology parameter n, the above Hamiltonian is *exactly solvable*. Moreover, the potential is also independent of r; hence we have constructed a single exactly solvable Hamiltonian which is associated to an infinite number of inequivalent Lie algebras of arbitrarily large dimension.

7   Matrix Schrödinger operators

We shall outline in this Section how to extend the notion of quasi-exact solvability to matrix Hamiltonians in one dimension. By definition, a *matrix Schrödinger operator* (or *matrix Hamiltonian*) is a $N \times N$ matrix of second-order differential operators of the form $\mathcal{H} = -\partial_x^2 + V(x)$, where $V(x)$ is a $N \times N$ Hermitian matrix of functions. If we drop the restriction that V be Hermitian, then $\mathcal{H}$ will be called a *Schrödinger-like* operator. Here we shall be mainly interested in spin $1/2$ matrix Hamiltonians, that is in the case $N = 2$; see [10] for the case of arbitrary N.

The notion of equivalence we shall use for matrix differential operators is the same as in the scalar case (cf. Definition 1), where the gauge factor $e^{\sigma(x)}$ is replaced now by an arbitrary invertible matrix $U(x)$. Suppose, as in the previous sections, that a Schrödinger operator

$$\mathcal{H}(x) = U(z) \cdot T(z) \cdot U(z)^{-1}, \qquad x = \varphi(z), \tag{43}$$

is equivalent to a matrix differential operator $T(z)$ preserving a finite-dimensional subspace $\mathcal{N} \subset \mathbb{C}^\infty(\mathbb{R}) \oplus \mathbb{C}^\infty(\mathbb{R})$ of smooth two-component wave functions, i.e., $T(\mathcal{N}) \subset \mathcal{N}$. It follows that $\mathcal{H}$ will restrict to a linear operator in the finite-dimensional space $U \cdot \mathcal{N}|_{z=\varphi^{-1}(x)}$, and therefore $\dim \mathcal{N}$ eigenfunctions of $\mathcal{H}$ will be computable by purely

algebraic methods. We shall say that $\mathcal{H}$ is *quasi-exactly solvable;* as in the scalar case, the algebraic eigenfunctions of $\mathcal{H}$ should obey appropriate boundary conditions, like for instance square integrability.

Thus, to find examples of quasi-exactly solvable matrix Hamiltonians we have to solve the following two problems:

1. Characterize all second-order matrix differential operators leaving invariant a finite-dimensional space of smooth functions

2. Find when a second-order matrix differential operator satisfying the previous condition is equivalent to a matrix Schrödinger operator

The first of these problems is clearly too general. In the scalar case, it is replaced by the simpler (but still highly nontrivial) problem of constructing second-order differential operators belonging to the enveloping algebra of a Lie algebra of quasi-exactly solvable first order differential operators, which in turn leads to the classification of such Lie algebras. In the matrix case this is still too general, since Lie superalgebras and even more general algebraic structures, whose complete classification is not available yet, come into play, [1], [2], [10].

To simplify matters, the idea is to *fix* a concrete (but still general enough) finite-dimensional subspace of smooth wave functions and classify all second-order matrix differential operators leaving this subspace invariant. Following Turbiner, [38], and Brihaye *et al.,* [1], [2], we choose as our invariant subspace the space of all wave functions with polynomial components. More precisely, let

$$\mathcal{P}_{m,n} = \mathcal{P}_m \oplus \mathcal{P}_n \tag{44}$$

be the vector space of wave functions $\Psi(z) = (\psi_1(z), \psi_2(z))^t$, with $\psi_1(z) \in \mathcal{P}_m$ and $\psi_2(z) \in \mathcal{P}_n$ polynomials of degrees less than or equal to m and n, respectively. We shall say that a matrix differential operator T is a *polynomial vector space preserving* (PVSP) operator if it preserves the vector space $\mathcal{P}_{m,n}$ for some non-negative integers m and n. We shall also denote by $\mathcal{P}_{m,n}^{(k)}$ the vector space of all PVSP matrix differential operators of differential order less than or equal to k. The problem is then to characterize all second-order PVSP operators, or in other words to describe the space $\mathcal{P}_{m,n}^{(2)}$ in as concise a way as possible.

To this end, let $\Delta = n - m \geq 0$, and consider the 2×2 differential operators

$$T^+ = \operatorname{diag}(J_{n-\Delta}^+, J_n^+), \quad T^0 = \operatorname{diag}(J_{n-\Delta}^0, J_n^0),$$
$$T^- = \operatorname{diag}(J^-, J^-), \quad J = \tfrac{1}{2}\operatorname{diag}(n+\Delta, n);$$
$$Q_\alpha = z^\alpha \sigma^-, \qquad \overline{Q}_\alpha = \bar{q}_\alpha(n, \Delta)\, \sigma^+, \quad \alpha = 0, \dots, \Delta, \tag{45}$$

with[†]

$$\bar{q}_\alpha(n, \Delta) = \prod_{k=1}^{\Delta-\alpha} (z\partial_z - n + \Delta - k)\, \partial_z^\alpha \quad \text{and} \quad \sigma^+ = (\sigma^-)^t = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{46}$$

[†]Here, and in what follows, we have adopted the convention that a product with its lower limit greater than the upper one is automatically 1.

Note that, in contrast with the scalar case, for $\Delta > 1$ the operators $\overline{Q}_\alpha$ have differential order grater than one.

It can be easily checked that the $6 + 2\Delta$ operators in (45) preserve $\mathcal{P}_{n-\Delta,n}$, as does any any polynomial in the above operators. We now introduce a $\mathbb{Z}_2$-grading in the set $\mathcal{D}_{2\times2}$ of 2×2 matrix differential operators as follows: an operator

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{47}$$

where a, b, c and d are differential operators, is even if $b = c = 0$, and odd if $a = d = 0$. Therefore, the T's and J are even and the Q's and $\overline{Q}$'s odd. This grading, combined with the usual product (composition) of operators, endows $\mathcal{D}_{2\times2}$ with an associative superalgebra structure. We can also construct a Lie superalgebra in $\mathcal{D}_{2\times2}$ by defining a generalized Lie product in the usual way:

$$[A, B]_s = AB - (-1)^{\deg A \deg B} BA. \tag{48}$$

However, this product does not close within the vector space spanned by the operators in (45), except for $\Delta = 0, 1$ (see [10] for the explicit commutation relations). More precisely, for $\Delta = 1$ the underlying algebraic structure is the classical simple Lie superalgebra $\mathfrak{osp}(2,2)$, [31], [38], whereas for $\Delta = 0$ it is $\mathfrak{h}_1 \oplus \mathfrak{sl}(2)$, where $\mathfrak{h}_1$ is the 3-dimensional Heisenberg superalgebra. As remarked in [1], in the latter case we can alternatively leave the grading aside and replace $J = 1$ by $\tilde{J} = \sigma_3$, ending up with the Lie algebra $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. For all other values of Δ, the Lie superalgebra of $\mathcal{D}_{2\times2}$ generated by the operators (45) is infinite-dimensional, [10].

In spite of the fact that in general they don't generate a finite-dimensional superalgebra, the operators (45) are the building blocks in the construction of second-order matrix differential operators preserving $\mathcal{P}_{m,n}$; in fact, any such operator must be a polynomial in the operators (45), a fact first stated by Turbiner, [38]:

Theorem 7 *Let $\Delta = n - m \geq 0$, $m \geq 2$, and let T be a second-order 2×2 matrix differential operator leaving the space $\mathcal{P}_{m,n}$ invariant. Then*

$$T = p_2(T^\epsilon) + J\,\tilde{p}_2(T^\epsilon) + \sum_{\alpha=0}^\Delta \overline{Q}_\alpha\,\bar{p}_{2-\Delta}^\alpha(T^\epsilon) + \sum_{\alpha=0}^\Delta Q_\alpha\,p_2^\alpha(T^\epsilon), \tag{49}$$

with p_k, $\tilde{p}_k$, p_k^α, and $\bar{p}_k^\alpha$ polynomials of degree less than or equal to k. If $\Delta = 0$, the above formula is still valid with J replaced by $\tilde{J} = \sigma_3$, while if $\Delta > 2$ every $\bar{p}_k^\alpha$ must be identically zero.

See [10] for a proof, as well as for the generalization of this result to the case of $N \times N$ matrix differential operators of arbitrary order.

Turning now to the second of the two fundamental problems listed at the begininng of this section, we have the following fundamental Theorem, [10]:

Theorem 8 *Let T be a PVSP operator in $\mathcal{P}^{(2)}_{m,n}$, with $n \geq m \geq 2$. Then T is equivalent to a Schrödinger-like operator if and only if it is of the form*

$$- T = p_4(z)\, \partial_z^2 + A_1(z)\, \partial_z + A_0(z)\,, \tag{50}$$

with $p_4(z)$ a polynomial of degree at most 4. The operator T is equivalent to a Schrödinger operator $-\partial_x^2 + V(x)$ if and only if (50) holds, and in addition we have:

1. *The physical coordinate $x = \varphi(z)$ is given by*

$$x = \int^z \frac{ds}{\sqrt{p_4(s)}}; \tag{51}$$

2. *There is an invertible matrix $\tilde{U}(x)$ satisfying the differential equation*

$$\tilde{U}_x = \tilde{U} A, \tag{52}$$

   with A given by[†] $A(x) = \left. \frac{1}{2\sqrt{p_4}}(A_1 - \frac{1}{2}p_4')\right|_{z=\varphi^{-1}(x)};$

3. *The matrix*

$$V = \tilde{U}\tilde{W}\tilde{U}^{-1}\,, \tag{53}$$

   with $\tilde{W} = -A_0 \circ \varphi^{-1} + A^2 + A_x$, is Hermitian.

The eigenfunctions of the Hamiltonian H are of the form $\Psi(x) = \tilde{U}(x)\Phi\left(\varphi^{-1}(x)\right)$, where $\Phi(z)$ is an eigenfunction of T and Ψ must satisfy appropriate boundary conditions.

Using the previous theorem, it can be shown that a matrix Schrödinger operator equivalent to a PVSP operator $T \in \mathcal{P}^{(2)}_{n-\Delta,n}$ with $\Delta > 1$ can be transformed by a constant gauge transformation into a Schrödinger operator whose potential matrix is triangular. Since triangular potentials correspond to essentially uncoupled systems, we can restrict ourselves without loss of generality to the cases $\Delta = 0$ and $\Delta = 1$.

The above theorems tell us, in principle, how to find examples of quasi-exactly solvable matrix Hamiltonians possessing a finite number of algebraically computable wave functions. The steps in the construction can be summarized as follows:

1. Construct the most general PVSP operator of the form (49). This operator will depend on a finite number of parameters, namely the coefficients of the arbitrary polynomials appearing in (49)

2. Compute the indefinite integral (51) to express the physical coordinate x in terms of the "canonical coordinate" z

3. Solve the matrix linear differential equation (52) to compute the gauge factor $\tilde{U}(x)$

[†]Here and in what follows, a subscripted x denotes derivation with respect to x, while derivatives with respect to z will be denoted with a prime $'$.

4. Compute the potential matrix $V(x)$ using equation (53)

5. Choose the values of the free parameters in such a way that the matrix $V(x)$ computed in the previous step is Hermitian, and that the phyisical wave functions satisfy the appropriate boundary conditions

Of the previous five steps, only the first two present practical problems. Indeed, the first step requires the computation and inversion of an elliptic integral, while the second one involves the solution of a linear matrix differential equation with variable coefficients.

The first of the latter two problems can be solved, as in the scalar case, by noting that there is still a "residual" $GL(2, \mathbb{R})$ symmetry which can be used to put p_4 into one of the canonical forms (26), for which the elliptic integral (51) reduces to an elementary function. We refer the interested reader to [10] for the complete story. As to the second problem (the solution of the matrix differential equation (52)), the strategy is to impose additional conditions on the matrix A that enable the explicit integration of (52), without being overly restrictive.

This can be done as follows. First of all, let us rewrite (52) in the z variable as

$$U' = U\hat{A}, \tag{54}$$

where

$$\hat{A}(z) = \frac{A(\varphi(z))}{\sqrt{p_4}} = \frac{1}{2\,p_4}(A_1 - \frac{1}{2}p_4'), \qquad U(z) = \tilde{U}(\varphi(z)). \tag{55}$$

Suppose that $\hat{A}$ satisfies

$$[\hat{A}(z), \int_{z_0}^z \hat{A}(s)\,ds] = 0\,, \tag{56}$$

for some $z_0 \in \mathbb{R}$. If this condition holds, we shall say that we are in the *commuting case*. (It can be shown that (56) is indeed verified by the quasi-exactly solvable Schrödinger operator found by Shifman and Turbiner, [33].) In the commuting case, we can readily integrate the gauge equation (54), obtaining the following general solution:

$$U(z) = U_0 \exp \int_{z_0}^z \hat{A}(s)\,ds, \qquad \text{where} \qquad U_0 \in GL(2, \mathbb{C}). \tag{57}$$

It can be easily shown that the most general 2×2 matrix $M(z)$ satisfying (56) must be of the form

$$M(z) = f(z)\,M_0 + g(z), \tag{58}$$

with M_0 a constant matrix and f, g scalar functions. Using this simple observation and the explicit formulas for the operators (45) for the cases $\Delta = 0, 1$, one readily computes the matrix $\hat{A}(z)$ corresponding to each of the canonical forms of $p_4(z)$. It turns out that the first three canonical forms (26) of p_4 lead to uncoupled Hamiltonians, and therefore can be discarded without loss of generality; see [10] for the explicit formulas for the remaining canonical forms.

In the *non*-commuting case (that is, when $[\hat{A}, \int^z \hat{A}] \neq 0$), we may still be able to integrate the gauge equation (54) explicitly by imposing other constraints on $\hat{A}$, as e.g., assuming it is uncoupled. Unfortunately, however, no interesting examples of quasi-exactly solvable Hamiltonians have been found so far in the non-commuting case.

Once a solution of the gauge equation (54) has been found, the potential matrix $V(x)$ can be immediately computed using (53). There are still two conditions that we must impose: first, that the potential matrix $V(x)$ in (53) be Hermitian, and secondly, that the algebraic wave functions satisfy suitable boundary conditions, which in the examples that follow simply reduce to square integrability. These conditions (particularly the Hermiticity of V) are still too complicated to be solved in full generality, even in the commuting case. This is completely analogous to what happened in the two-dimensional scalar case (Section 6), where the closure conditions cannot be solved in general, and we are therefore restricted to obtaining particular solutions. In the same spirit, we shall exhibit now two examples of quasi-exactly solvable spin $1/2$ matrix Hamiltonians, all of which belong to the commuting case described above; see [10] for additional examples and further details. Interestingly, both of these examples correspond to the case $\Delta = 1$; indeed, in the commuting case with $\Delta = 0$ every potential we have obtained turned out to be either non-normalizable, singular or diagonalizable by a constant gauge transformation.

For our first example we consider the PVSP operator

$$
\begin{aligned}
-T = {} & (T^0)^2 + 2a_2 T^+ + 2(n+1)T^0 - 2\, JT^0 + 2b_1\, \overline{Q}_0 \\
& - 2b_1\, Q_0 T^0 - (3n+1)b_1\, Q_0 - 4a_2 b_1 Q_1 - (2\hat{d}_0 + n + \frac{1}{2})J \,,
\end{aligned}
\tag{59}
$$

with all the parameters real. In this example $p_4 = z^2$, so that we are in case 6 of Theorem 6. Solving equation (51) for z we immediately obtain $z = e^x$. The gauge factor reads:

$$
U(z) = \sqrt{z}\, e^{a_2 z}
\begin{pmatrix}
\cos u & \sin u \\
-\sin u & \cos u
\end{pmatrix},
\qquad \text{where} \qquad u = b_1 \log z \,.
\tag{60}
$$

The potential is given by

$$
\begin{aligned}
v_j &= -(2n+1)a_2\, e^x + a_2^2\, e^{2x} + (-1)^j(\alpha(x)\cos(2b_1 x) - \beta(x)\sin(2b_1 x)), \\
v &= \alpha(x)\sin(2b_1 x) + \beta(x)\cos(2b_1 x) \,,
\end{aligned}
\tag{61}
$$

where $j = 1, 2$, and

$$
\alpha(x) = -\frac{\hat{d}_0}{2} + a_2\, e^x \,, \qquad \beta(x) = (2n+1)\, b_1 + 2a_2 b_1\, e^x \,.
\tag{62}
$$

It is easily verified that the expected value of the potential is bounded from below, i.e.,

$$
\langle \Psi, V\Psi \rangle \geq c\, \|\Psi\|^2 \,, \qquad \text{with} \quad \Psi \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \,,
\tag{63}
$$

for some $c \in \mathbb{R}$. (Note, however, that even in this case the amplitude of the oscillations of $v(x)$ tends to infinity as $x \to +\infty$.) Finally, the condition $a_2 < 0$ is necessary and sufficient to ensure the square integrability of the eigenfunctions $\Psi(x)$.

As our last example, we consider:

$$-T = T^- T^0 + 2a_1 T^0 + \left(2a_0 + n - \frac{1}{2}\right) T^- - JT^- + 2b_1 \overline{Q}_0 - 2b_1 Q_0 T^0$$
$$- b_1 \left(4a_0 + 3n + 1\right) Q_0 - 4a_1 b_1 Q_1 + 2(2\hat{a}_0 - a_1) J \,, \qquad (64)$$

where all the coefficients are real, and $b_1 \neq 0$. Since $p_4 = z$ (case 7), we have $z = x^2/4$. The gauge factor is chosen as follows:

$$U(z) = z^{a_0} \, e^{a_1 z} \begin{pmatrix} \cos b_1 z & \sin b_1 z \\ -\sin b_1 z & \cos b_1 z \end{pmatrix} \,.$$

The entries of the potential $V(x)$ are given by

$$\begin{aligned} v_j &= \frac{2a_0(2a_0 - 1)}{x^2} + \frac{1}{4}\left(a_1^2 - b_1^2\right) x^2 + (-1)^j (\hat{a}_0 \, \cos \frac{b_1 x^2}{2} - \alpha(x) \, \sin \frac{b_1 x^2}{2}) \,, \\ v &= \hat{a}_0 \, \sin \frac{b_1 x^2}{2} + \alpha(x) \, \cos \frac{b_1 x^2}{2} \,, \end{aligned} \qquad (65)$$

with $j = 1, 2$, and $\alpha(x)$ is defined by

$$\alpha(x) = \frac{b_1}{2}\left(4a_0 + 4n + 1 + a_1 x^2\right) \,.$$

We first note that the potential is singular at the origin unless $a_0 = 0, 1/2$. Let us introduce the parameter $\lambda = 2a_0 - 1$, in terms of which the coefficient of x^{-2} in v_j is $\lambda(\lambda + 1)$. If λ is a non-negative integer l, we may regard

$$(-\partial_x^2 + V(x) - E)\Phi(x) = 0 \,, \qquad 0 < x < \infty \,, \qquad (66)$$

as the radial equation obtained after separating variables in the three-dimensional Schröd-inger equation with a spherically symmetric Hamiltonian given by

$$\hat{H} = -\Delta + U(r) \,, \qquad \text{with} \qquad U(x) = V(x) - \frac{l(l+1)}{x^2} \,,$$

where Δ denotes the usual flat Laplacian. Given a non-negative integer l and a spherical harmonic $Y_{lm}(\theta, \phi)$, if Φ is an eigenfunction for the equation (66) satisfying

$$\lim_{x \to 0^+} \Phi(x) = 0 \,, \qquad (67)$$

then

$$\hat{\Psi}(r, \theta, \phi) = \frac{\Phi(r)}{r} Y_{lm}(\theta, \phi)$$

will be an eigenfunction for $\hat{H}$ with angular momentum l. If λ is not a non-negative integer, we shall consider (66) as the radial equation for the singular potential $U(r) = V(r)$ at zero angular momentum. The potential $U(r)$ is physically meaningful, in the sense that the Hamiltonian $\hat{H}$ admits self-adjoint extensions and its spectrum is bounded from

below, whenever $\lambda \neq -1/2$, [11], [17]. The boundary condition (67) must be satisfied in the singular case for *all* values of λ. This boundary condition is verified if and only if $a_0 > 0$. The expected value of the potential is bounded from below, that is, equation (63) holds, if and only if

$$\left| \frac{a_1}{b_1} \right| > 1 + \sqrt{2} \,.$$

Finally, the conditions

$$a_0 \geq 0 \,, \qquad a_1 < 0 \,,$$

guarantee the square integrability of the eigenfunctions $\Phi(x)$.

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